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Exotic states in the cores of quantised vortices for superfluids and superconductors

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Abstract. We consider the core structures of quantised vortex lines in Bose and Fermi superfluid systems. We emphasise the result that, within the core of a singly quantised vortex line in a Bose superfluid, there spontaneously appears a condensate of boson pairs in a relative d-wave state, instead of the normal fluid. Also, in the case of the singly quantised Abrikosov vortex line in a superconductor, instead of normal electrons there emerges a novel state of superconductivity, in which a condensate of bosons made of four fermions appears. In particular, we discuss these core structures as further examples for the flaring-out of the vortex singularity into higher dimensions, in analogy with the extended-space topology of the superfluid ^{3}He-B vortex core; we also relate this generic behaviour to the Laughlin state within the quantum Hall effect.

1. Introduction

Recently, in the context of high-T_c superconductivity, the fractionally quantised Hall effect (QHE) and heavy-fermion metals, a new impetus has been given to a search for exotic many-body ground states—both for fermions and for bosons—such as the Laughlin ground state in the fractional QHE (Laughlin 1987), the resonant valence bond state in the Hubbard model (Liang *et al* 1988), their combination (Laughlin 1988), new variational functions for strongly correlated fermions (Bouchaud *et al* 1988), and pair condensation in Bose systems, to name but a few examples.

We propose that at least some of the new ground states may exist inside the cores of quantised vortices in ordinary Bose-condensate superfluids, such as He II, or in the cores of Abrikosov vortices in conventional superconductors, assuming that the vortex-core radius is sufficiently greater than the inter-atomic distance. This is normally satisfied in superconductors having a large coherence length ξ, but for He II it holds only in the vicinity of T_c, where $\xi(T)$ becomes long enough.

Inside the core of a conventional singly quantised vortex in a conventional superfluid (superconductor), the conventional superfluidity (superconductivity) is suppressed; the order parameter (below, (r, φ, z) denote the cylindrical coordinates):

$$\psi_v = C(r)\exp(i\varphi) \tag{1.1}$$

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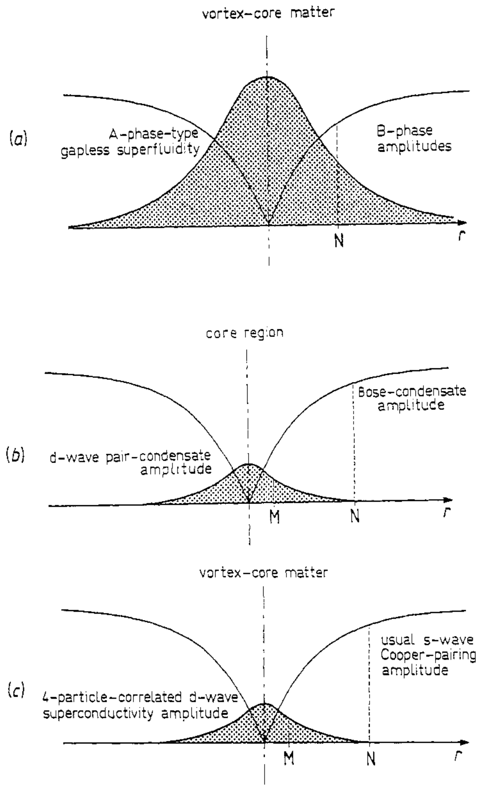

Figure 1. (a) Schematic illustration for the hard-core structure of a singly quantised vortex in superfluid ^{3}He-B. The real-space zero of the B-phase vortex dissipates into the point vortices (gap nodes) on the Fermi sphere in the A-phase-type superfluidity of the vortex-core matter (for full details, see Salomaa and Volovik (1987)). (b) The core structure of a singly quantised vortex in a Bose superfluid exhibits, in addition to the bose-condensate amplitude $\langle\hat{\psi}\rangle$ which vanishes on the vortex axis, a d-wave pair-condensate amplitude concentrated in the hard core of the vortex. The real-space zero on the vortex axis dissipates into zeros in the relative space of atoms comprising a d-wave pairing state. (c) The singly quantised Abrikosov vortex in a conventional superconductor displays exotic four-particle-correlated superconductivity in the vortex-core matter, where the usual Cooper-pairing amplitude tends to zero. The real-space zero also dissipates into zeros in the relative space of four atoms, which form the boson with the internal orbital momentum $L^{\text{int}} = 2$. In (a), (b) and (c), points N and M indicate the distances $R_1 = \xi_0(1 - T/T_c)^{-1/2}$ and $R_2 = \xi_0(1 - T/T_c)$, respectively.

tends to zero on the vortex axis, $C(r = 0) = 0$, owing to the winding of the phase by 2π around the vortex line. The normal-fluid state in the vortex core proves, however, to be unstable and there arise novel states which are intermediate between a normal fluid and the conventional superfluid system.

An example of this kind is provided by the ^{3}He-B vortex (for a review, see Salomaa and Volovik (1987)). Superfluid ^{3}He-B may be considered a conventional Fermi superfluid in the sense that the superfluid energy gap is finite everywhere on the Fermi surface. Inside the core of this vortex, superfluidity is not broken; instead, the ^{3}He-A state appears (figure 1(a)), which may be regarded as an intermediate stage between normal liquid ^{3}He and the conventional superfluid ^{3}He-B; the ^{3}He-A superfluidity is gapless, with the gap turning to zero at point nodes on the Fermi surface.

The most important feature in the appearance of the intermediate states in the quantised vortices is the flaring-out of zeros from the real (r) space; in the example on ^{3}He-B, the zero in the order parameter at $r = 0$ in the position space transforms into the zeroes of the gap ('point vortices on the Fermi sphere') in the momentum space. The vortex zero is of topological origin owing to the winding of the phase; it cannot disappear but it may, nevertheless, transform into extra dimensions, e.g. into the momentum ($\boldsymbol{k}$) space, thus producing—by necessity—intermediate states of superfluidity in the vortex-core matter.

Another example is provided by the flaring-out of the zeros into the k-space in a doubly quantised Abrikosov vortex; inside the core of this vortex, a superconducting d-wave condensate should appear, with zeros in the superconducting gap on the Fermi surface (Volovik 1988). Here we introduce two further examples of exotic ground states within the core matter of quantised vortex lines.

(i) In the core of the singly quantised vortex in a Bose superfluid, there appears a condensate of boson pairs in a relative d-wave state, instead of the normal liquid on the vortex axis (see figure 1(b)).

(ii) In the core of the singly quantised Abrikosov vortex in a superconductor, there exists a novel correlated superconducting four-particle state with the orbital-momentum projection $L_z = 2$ for the composite boson formed by the four fermions (see figure 1(c)).

We discuss how the flaring-out of zeros into extra dimensions takes place in this case and how the scenario is connected with the Laughlin state in the QHE. The appearance of the new correlated states in the cores of singly quantised vortices in the superfluid Bose condensate and in superconductors may be seen from symmetry arguments and also with use of the Ginzburg–Landau free-energy functional.

2. Symmetry considerations

Here it is shown that symmetry dictates the existence of an exotic pairing state in the core of a superfluid vortex. Let us begin with the singly quantised vortex in the superfluid Bose liquid—such as He II—in which the order parameter, the mean value of the second-quantised annihilation operator

$$\psi = \langle \hat{\psi} \rangle \tag{2.1}$$

has the form $\psi = \psi_v$ in equation (1.1). The vortex state (1.1), with real $C(r)$, has the following symmetries.

(i) A continuous symmetry described by the generator

$$\hat{Q} = \hat{L}_z - \hat{I} \tag{2.2}$$

which means the axisymmetry of the vortex line. Here $\hat{L}_z$ is the total orbital-momentum operator, $\hat{L}_z = \hat{L}_z^{\text{ext}} + \hat{L}_z^{\text{int}}$, including the internal angular momentum of the particles comprising the boson in the Bose condensate, and the angular momentum for the centre-of-mass motion of the particles, while $\hat{I}$ is the number operator for the Bose particles, which serves as the generator of the gauge transformation: $\hat{U}(\Phi) = \exp(\mathrm{i}\hat{I}\Phi)$.

The action of this operator on ψ is as follows:

$$\hat{L}_z^{\text{int}} \psi = 0 \tag{2.3}$$

since the bosons in conventional Bose superfluids have no internal degrees of freedom, while

$$\hat{L}_z^{\mathrm{ext}}\psi = (1/\mathrm{i})(\partial/\partial\varphi)\psi \tag{2.4}$$

since the centre-of-mass motion just means the particle motion. Also,

$$\hat{I}\psi = \psi \tag{2.5a}$$

$$\hat{U}(\Phi)\psi = \exp(\mathrm{i}\hat{I}\Phi)\,\psi = \psi\exp(\mathrm{i}\Phi) \tag{2.5b}$$

since the boson in the Bose condensate consists of a single particle, $I = 1$.

According to equations (2.2)–(2.5), we have for the vortex state ψ_v of equation (1.1) $\hat{L}_z^{\mathrm{ext}}\psi_\mathrm{v} = \psi_\mathrm{v}$, $\hat{L}_z^{\mathrm{int}}\psi_\mathrm{v} = 0$ and $\hat{I}\psi_\mathrm{v} = \psi_\mathrm{v}$; consequently, one finds that

$$\hat{Q}\psi_\mathrm{v} = 0. \tag{2.6}$$

In other words, there exists a continuous transformation

$$\exp(\mathrm{i}\hat{Q}\alpha)\,\psi_\mathrm{v} = \psi_\mathrm{v} \tag{2.7}$$

with an arbitrary α which does not change the order parameter of the vortex state (1.1). This is just the continuous symmetry of the quantised vortex line.

(ii) The discrete symmetry $\hat{P}\hat{U}(\pi)$, where $\hat{P}$ is the space-inversion operation

$$\hat{P}\psi(\boldsymbol{r}) = \psi(-\boldsymbol{r}) \tag{2.8}$$

and $\hat{U}(\pi)$ denotes the gauge transformation through the phase $\Phi = \pi$:

$$\exp(\mathrm{i}\hat{I}\pi)\,\psi(\boldsymbol{r}) = -\psi(\boldsymbol{r}) \tag{2.9}$$

according to equation (2.5).

For the vortex state of equation (1.1), one can check that this combination leaves the vortex state invariant, i.e.

$$\hat{P}\hat{U}(\pi)\psi_\mathrm{v} = \psi_\mathrm{v} \tag{2.10}$$

is obeyed.

(iii) There exists yet a further combined discrete symmetry, $\hat{T}\hat{O}_{x,\pi}$, where $\hat{T}$ is the time-inversion operation

$$\hat{T}\psi = \psi^* \tag{2.11}$$

and $\hat{O}_{x,\pi}$ is the space rotation about the axis $\hat{x}$ by the angle π:

$$\hat{O}_{x,\pi}\psi(\boldsymbol{r}) = \psi(\hat{O}_{x,\pi}\boldsymbol{r}). \tag{2.12}$$

For the vortex state in equation (1.1), one has

$$\hat{T}\hat{O}_{x,\pi}\psi_\mathrm{v} = \psi_\mathrm{v} \tag{2.13}$$

since $C(r)$ is real.

The order parameter $\psi_\mathrm{v}(r)$ is zero at the origin, since ψ_v is an eigenstate of the operator $\hat{L}_z^{\mathrm{ext}}$ (i.e. $\hat{L}_z^{\mathrm{ext}}\psi_\mathrm{v} = \psi_\mathrm{v}$), with the non-vanishing eigenvalue $L_z^{\mathrm{ext}} = 1$. This implies that the superfluidity of the single-particle condensate is broken on the vortex axis. It proves interesting to consider whether some other superfluid state could arise in the vortex core. This may occur, provided that the following two conditions are fulfilled.

(i) The state should have the same symmetries (2.7), (2.10) and (2.13) as the original vortex state.

(ii) It should have vanishing centre-of-mass orbital momentum: $\hat{L}_z^{\rm ext} = 0$.

The pair-condensate superfluid state, which is the eigenstate of $\hat{I}$ with the eigenvalue $I = 2$, is a correct candidate if it possesses the internal orbital momentum $L_z^{\rm int} = 2$ for the 'Cooper' pairs, and no φ-dependence of the centre-of-mass coordinate, i.e. $\hat{L}_z^{\rm ext} = 0$; in this case $\hat{Q} = \hat{I} - \hat{L}_z^{\rm int} = 0$, and the first symmetry equation (2.7) is satisfied. In order to fulfil also the symmetry (2.10), these 'Cooper' pairs should display an even internal momentum; therefore $L^{\rm int} = 2$, the minimal possible, i.e. a d-wave pairing state occurs inside the core. Thus the order parameter for the pair-correlated state inside the core is a traceless symmetric tensor B_{ij}, which transforms under a gauge transformation as

$$\hat{U}(\Phi)B_{ij} = B_{ij}\exp(2\mathrm{i}\Phi) \tag{2.14}$$

and it is related to the boson pair-condensate 'Gor'kov' function in the following way:

$$F(\boldsymbol{r}_1, \boldsymbol{r}_2) = \langle\hat{\psi}(\boldsymbol{r}_1)\hat{\psi}(\boldsymbol{r}_2)\rangle = \langle\hat{\psi}(\boldsymbol{r}_1)\rangle\langle\hat{\psi}(\boldsymbol{r}_2)\rangle + B_{ij}(\boldsymbol{r})(\rho_i\rho_j - \tfrac{1}{3}\delta_{ij}\rho^2)f(\boldsymbol{\rho}^2, (\boldsymbol{\rho}\cdot\hat{z})^2) \tag{2.15}$$

where the centre-of-mass and the relative coordinates are given, respectively, by

$$\boldsymbol{r} = \tfrac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2) \qquad \boldsymbol{\rho} = \boldsymbol{r}_1 - \boldsymbol{r}_2.$$

The first term on the RHS of equation (2.15) arises from the ordinary single-particle Bose condensate, $\langle\hat{\psi}\rangle = \psi$.

The order parameter $B_{ij}(\boldsymbol{r})$ depends on the centre-of-mass coordinate $\boldsymbol{r}$, and for the state with $L_z^{\rm int} = 2$, $L_z^{\rm ext} = 0$ in the vortex it has the form

$$B_{ij}^{\rm v}(\boldsymbol{r}) = C_{22}(r)(\hat{x}_i + \mathrm{i}\hat{y}_i)(\hat{x}_j + \mathrm{i}\hat{y}_j) \tag{2.16}$$

where $\hat{x}$ and $\hat{y}$ are Cartesian coordinates in the plane transverse to the vortex axis ($\hat{z}$).

The $\hat{T}\hat{O}_{x,\pi}$ symmetry requires $C_{22}(r)$ to be real. Now, since the pair-correlated state (2.16) has the same symmetry as the single-particle state (1.1), i.e.

$$\hat{Q}B_{ij}^{\rm v} = 0 \tag{2.17a}$$

owing to

$$\hat{L}_z^{\rm int}B_{ij}^{\rm v} = 2B_{ij}^{\rm v} \qquad \hat{L}_z^{\rm ext}B_{ij}^{\rm v} = 0 \qquad \hat{I}B_{ij}^{\rm v} = 2B_{ij}^{\rm v} \tag{2.17b}$$

$$\hat{T}\hat{O}_{x,\pi}B_{ij}^{\rm v} = \hat{P}\exp(\mathrm{i}\hat{I}\pi)\,B_{ij}^{\rm v} = B_{ij}^{\rm v} \tag{2.17c}$$

it should, therefore, necessarily be present in the vortex and even dominate near the vortex core, where the ordinary superfluidity is suppressed since there is no phase winding in (2.16) which would force $C_{22}(r = 0)$ be be zero; the precise form of $C_{22}(r)$ may be found easily in the Ginzburg–Landau region.

3. Ginzburg–Landau functional for a mixture of single-particle and pair Bose condensates

The free-energy functional which describes the appearance of the Bose pair condensate in the core of a quantised vortex of the conventional single-particle Bose condensate is

$$F = F_1 + F_2 + F_{12} \tag{3.1}$$

where F_1 is the usual Ginzburg–Landau functional for the order parameter ψ of the conventional single-particle condensate:

$$F_1 = -\alpha(\tau)|\psi|^2 + \tfrac{1}{2}\beta|\psi|^4 + \gamma|\nabla\psi|^2 \tag{3.2}$$

where

$$\alpha(\tau) = \alpha_0 \tau \qquad \tau = 1 - T/T_c \tag{3.3}$$

with

$$\alpha_0 > 0 \qquad \beta > 0 \qquad \gamma > 0.$$

The temperature-dependent superfluid coherence length is

$$\xi(\tau) = [\gamma/\alpha(\tau)]^{1/2} = \xi_0 \tau^{-1/2}. \tag{3.4}$$

The functional F_2 is the free energy of a d-wave pair Bose condensate. Since this state is unfavourable in the bulk homogeneous liquid, it has a positive quadratic term

$$F_2 = \eta B_{ij} B_{ij}^* \tag{3.5}$$

with $\eta > 0$; therefore, all other terms may be neglected. The free-energy contribution F_{12} in equation (3.1) describes the interaction of the single-particle and the pair condensate order parameters. The gauge symmetry (the gauge transformations of the fields ψ and B_{ij} are given by equations (2.5) and (2.14), respectively), the time-inversion symmetry and the rotational symmetry of the free energy enforce the following form upon F_{12}:

$$F_{12} = \varepsilon (B_{ij}^* \nabla_i \psi \nabla_j \psi + \mathrm{cc}). \tag{3.6}$$

All the parameters in the functional (3.1)—with the exception of $\alpha(\tau)$—i.e. β, γ, η and ε are temperature independent in the Ginzburg–Landau approximation.

The minimisation of F with respect to B_{ij} yields

$$B_{ij} = (\varepsilon/\eta) \nabla_i \psi \nabla_j \psi. \tag{3.7}$$

For the vortex state (1.1), this results in the following distribution of the d-wave pair condensate within the vortex:

$$B_{ij}^{\mathrm{v}} = C_{22}(r)(\hat{x} + \mathrm{i}\hat{y})_i (\hat{x} + \mathrm{i}\hat{y})_j + C_{20}(r) \exp(2\mathrm{i}\varphi)(\delta_{ij} - \hat{z}_i \hat{z}_j) + C_{2,-2}(r) \exp(4\mathrm{i}\varphi)(\hat{x} - \mathrm{i}\hat{y})_i (\hat{x} - \mathrm{i}\hat{y})_j \tag{3.8}$$

where the amplitude $C_{22}(r)$ of the d-wave pair state with momentum $L_z^{\mathrm{int}} = 2$ is given by

$$C_{22}(r) = \tfrac{1}{4}(\varepsilon/\eta)[C'(r) + C(r)/r]^2. \tag{3.9}$$

Above, the prime indicates a radial derivative: $C' = \partial C/\partial r$. The amplitude C_{20} for the pair state with $L^{\mathrm{int}} = 2$, $L_z^{\mathrm{int}} = 0$ is

$$C_{20}(r) = \tfrac{1}{2}(\varepsilon/\eta)\{[C'(r)]^2 - [C(r)/r]^2\} \tag{3.10}$$

while the amplitude $C_{2,-2}$ for the pair state with $L^{\mathrm{int}} = 2$ and $L_z^{\mathrm{int}} = -2$ equals

$$C_{2,-2}(r) = \tfrac{1}{4}(\varepsilon/\eta)[C'(r) - C(r)/r]^2. \tag{3.11}$$

Since $C(r)$ depends linearly on r at the origin, we find for $r \to 0$

$$C(r) \to a(\tau) r \tag{3.12}$$

with $a(\tau) = a_0 \tau$, and where

$$a_0 \sim \frac{1}{\xi_0} \sqrt{\alpha_0/\beta}.$$

Hence the amplitudes C_{20} and $C_{2,-2}$ of the states having a winding phase tend to zero on the vortex axis: $C_{20}(r=0) = C_{2,-2}(r=0) = 0$, while the amplitude C_{22} of the state with the quantum numbers $L^{\text{int}} = 2$ and $L_z^{\text{int}} = 2$ survives even in the vortex core:

$$C_{22}(r=0) = (\varepsilon/\eta)a^2 = (\varepsilon/\eta)a_0^2\tau^2. \tag{3.13}$$

Let us estimate the influence of the pair condensate, say, on the longitudinal component (along $\hat{z}$, the vortex line) of the superfluid density tensor, in comparison with that of the single-particle superfluid condensate density near the vortex axis:

$$\rho_s^{(1)} \sim \gamma|\psi|^2 \sim \beta r^2\tau^2. \tag{3.14}$$

Since $\varepsilon^2/\eta \sim \beta\xi_0^4$, one finds the following contribution to ρ_s from the pair condensate:

$$\rho_{s\|}^{(2)} \sim \beta\xi_0^2\tau^4. \tag{3.15}$$

The pair condensate superfluid density thus becomes dominant only for distances of the order of $r_0 \sim \xi_0\tau \ll \xi_0$ from the vortex axis, which is meaningless in practice for He II, where ξ_0 is of the order of the inter-atomic spacing; however, this length scale becomes relevant for the low-temperature superconductors with large coherence length ξ_0. What seems to be more important for He II is the internal orbital momentum of the boson pairs within the core; this may, in particular, result in an electronic orbital ferromagnetism for the ^{4}He atoms—induced by the orbital momentum L_z^{int} of the pair condensate—exactly in the same way as for ^{3}He-A (cf Leggett 1977, Paulson and Wheatley 1978).

Let us now turn to consider how the zeros in the order parameter dissipate in the core owing to the pair condensate.

4. Flaring-out of zeros inside the vortex core

In order to investigate the flaring-out of the vortex singularity, it is helpful to discuss the example of the pair function $F(\boldsymbol{r}_1, \boldsymbol{r}_2)$ in equation (2.15). Introducing the complex coordinate $Z = x + iy$, one may present $F(\boldsymbol{r}_1, \boldsymbol{r}_2)$ for the vortex in the following form:

$$\begin{aligned} F_v(\boldsymbol{r}_1, \boldsymbol{r}_2) = {} & \tilde{C}(|Z_1|)\tilde{C}(|Z_2|)Z_1Z_2 \\ & + C_{22}(|Z_1 + Z_2|/2)(Z_1 - Z_2)^2 f(|Z_1 - Z_2|^2, (z_1 - z_2)^2) \end{aligned} \tag{4.1}$$

where

$$\tilde{C}(|Z|) = C(r)/r$$

or, for the general case of the vortex located at the point $\zeta = x_v + iy_v$,

$$\begin{aligned} F_v(\boldsymbol{r}_1\, \boldsymbol{r}_2) = {} & \tilde{C}(|Z_1 - \zeta|)\tilde{C}(|Z_2 - \zeta|)(Z_1 - \zeta)(Z_2 - \zeta) \\ & + C_{22}[|(Z_1 + Z_2)/2 - \zeta|](Z_1 - Z_2)^2 f(|Z_1 - Z_2|^2, (z_1 - z_2)^2). \end{aligned} \tag{4.2}$$

Since $C(r)$ changes from a constant to zero on the path from infinity to the vortex axis, while in contrast $C_{22}(r)$ changes from zero to a constant value along the same path, one has a continuous transformation from the polynomial

$$P_v = (Z_1 - \zeta)(Z_2 - \zeta) \tag{4.3}$$

describing the zeros in the inhomogeneous vortex state, into the polynomial:

$$P_{\text{d-wave}} = (Z_1 - Z_2)^2 \tag{4.4}$$

representing the zeros in a homogeneous d-wave pairing state with the relative momentum $L_z^{\rm int} = 2$ inside the core.

In this transformation, the zeros of Z_1 and Z_2 at the position ζ of the vortex flare out from the real space into the zeros in the relative-position space. From the topological point of view, there thus occurs a deformation of the manifold of zeros of the function $F(\boldsymbol{r}_1, \boldsymbol{r}_2)$.

The zeros of the complex function

$$F(\boldsymbol{r}_1, \boldsymbol{r}_2) = |F(\boldsymbol{r}_1, \boldsymbol{r}_2)| \exp[\mathrm{i}\Phi(\boldsymbol{r}_1, \boldsymbol{r}_2)]$$

are topologically stable, since these form the singular manifold where the phase variable $\Phi(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is undefined, owing to its winding around the manifold. That is, these are singularities belonging to the homotopy group Π_1. In the four-dimensional (Z_1, Z_2)-space, the manifold of zeros is a two-dimensional singular surface. In the initial case of the polynomial in equation (4.3), this surface consists of two sheets, each with unit winding number ($n = 1$), i.e. firstly $Z_1 = \zeta$, with Z_2 arbitrary, and secondly $Z_2 = \zeta$, with arbitrary Z_1, while in the non-vanishing state of the polynomials (4.4) these sheets merge together, thus forming the singular surface with double winding of $\Phi(\boldsymbol{r}_1, \boldsymbol{r}_2)$, i.e. $n = 2$.

This is just another representation for the same flaring-out of vorticity that is familiar from the Fermi systems, the $(\boldsymbol{r}_1, \boldsymbol{r}_2)$-space, instead of the $(\boldsymbol{r}, \boldsymbol{k})$-space, with $\boldsymbol{k}$ on the Fermi surface, originating from the relative coordinate $\boldsymbol{r}_1 - \boldsymbol{r}_2$ in the Fourier transform, and $\boldsymbol{r} = (\boldsymbol{r}_1 + \boldsymbol{r}_2)/2$ being the centre-of-mass coordinate.

The flaring-out of the vorticity into the $\boldsymbol{k}$-space in the core of the doubly quantised Abrikosov vortex in a Fermi superfluid (cf Volovik 1988) may also be given in the (Z_1, Z_2)-representation; the pair function

$$F_{\alpha,\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2) = g_{\alpha\beta} F(\boldsymbol{r}_1, \boldsymbol{r}_2) \tag{4.5}$$

with

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is proportional to the polynomial

$$P_{\rm v} = [(Z_1 + Z_2)/2 - \zeta]^2 \tag{4.6}$$

outside the core, representing the double vortex in the centre-of-mass space, while in the core it transforms into the d-wave state in equation (4.4). This also means the reorientation of the two-dimensional manifold of zeros (figure 2) again with double winding of the phase, i.e. $n = 2$.

It proves instructive to write down the many-body particle wavefunction, describing the vortex state in a Bose superfluid:

$$\Psi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = \left(\sum_{\text{symmetrisation}} F(Z_1, Z_2) F(Z_3, Z_4) \ldots F(Z_{N-1}, Z_N) \right) \Psi_0(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \tag{4.7}$$

where Ψ_0 represents a function without topological zeros. Provided that the amplitude C_{22} of the boson-pair condensate were neglected, this would be just the conventional vortex state

$$\Psi_{\rm v} = (Z_1 - \zeta)(Z_2 - \zeta) \ldots (Z_N - \zeta) \Psi_0 \tag{4.8}$$

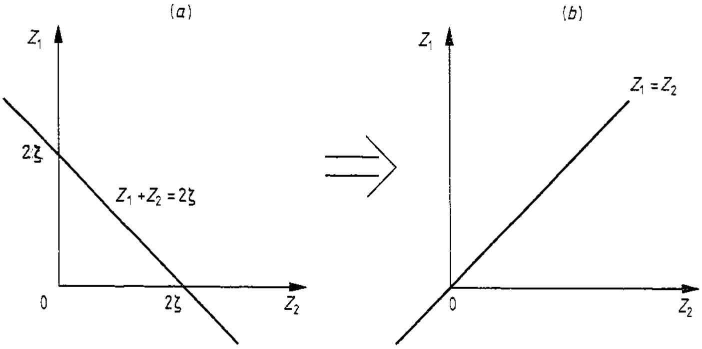

Figure 2. Reorientation of the manifold of topological zeros with the charge $n = 2$ in the process of flaring-out of vorticity from (*a*) the real-space singularity on the vortex axis of a doubly quantised Abrikosov vortex at the point $\zeta = x_v + iy_v$, described by

$$[(Z_1 + Z_2)/2 - \zeta]^2$$

into (*b*) the singularity in the relative-coordinate space, described through $(Z_1 - Z_2)^2$, which corresponds to a d-wave superconductivity state of the vortex-core matter with point nodes in the gap.

while in the opposite case, where the amplitude C for the conventional condensate is neglected, this would describe the homogeneous d-wave pairing state

$$\Psi_d = \left(\sum_{\text{symmetrisation}} (Z_1 - Z_2)^2 (Z_3 - Z_4)^2 \ldots (Z_{N-1} - Z_N)^2 \right) \Psi_0. \tag{4.9}$$

In the latter case, the transformation from the form (4.8) into equation (4.9) means the reorientation and deformation of the $(3N - 2)$-dimensional hypersurface of zeros of the complex function $\Psi(r_1, \ldots, r_N)$, defined in the $3N$-dimensional space.

Thus we have arrived at the classes of the many-body wavefunctions Ψ; the states which may be transformed to each other by the deformation of the manifold of topologically stable zeros, without changing the total topological charge (winding number), belong to the same class. In this sense, the Bose-superfluid state with one vortex is in the same topological class as the homogeneous pair-condensate state with $L_z^{\text{int}} = 2$. Note that the statement is in fact much more general than in the trivial case of the polynomial in Z_i, where the class is defined simply by the degree of the polynomial, the topological class is defined for a general type of wavefunction dependent on all the three spatial coordinates x_n, y_n, z_n for each particle n, with the dimension of the manifold of zeros being $3N - 2$.

The transformation from one class to another occurs if in the intermediate state the manifold of zeros with higher dimension ($3N - 1$ or $3N$) appears or if in the non-confined geometry some of the topologically charged zeros move over to infinity. As two examples of such transformations in the simplest possible case of a two-dimensional phase space (x, y), when the zero manifold is the system of points, consider

$$\text{I: } \psi(x, y) = x + \mathrm{i}y \tanh \alpha. \tag{4.10}$$

When α changes from $-\infty$ to ∞, the point zero with the topological charge $n = 1$ at $x = 0$, $y = 0$ transforms into the zero with charge $n = -1$ at the same point. The transition occurs at $\alpha = 0$, where the manifold of zeros becomes the whole line $x = 0$.

Secondly, consider

$$\text{II: } \psi(x, y) = Z - aZ^2. \tag{4.11}$$

Here there are two point zeros at $Z = 0$ and $Z = 1/a$, both with $n = 1$ and with the total charge 2; the second zero goes to infinity for $a \to 0$, thus leaving only unit charge in the system.

It is also instructive to consider the wavefunction of a Bose superfluid with many vortices. The state with K vortices and N particles

$$\Psi_{\mathrm{V}}^{(K,N)} = \prod_{\substack{a \leq K \\ n \leq N}} (Z_n - \zeta_a)\Psi_0 \tag{4.12}$$

is in the same class as the homogeneous Laughlin state for the fractional QHE for bosons, i.e.

$$\Psi_{\text{Laughlin}} = \prod_{i,j} (Z_i - Z_j)^q \Psi_0 \tag{4.13}$$

with q chosen even for bosons, if the relation

$$\tfrac{1}{2}(N - 1)q = K \tag{4.14}$$

is fulfilled, i.e. if the number of vortices is $q/2$ times larger than the number of particles in the system.

The continuous transition from equation (4.12) to equation (4.13), with the flaring-out of zeros from the points ζ_a in real space into the space of relative coordinates, occurs through the successive nucleation on the vortex axis of the pair-correlated states, four-particle correlated states, etc, until finally the N-particle correlated state in equation (4.13) is reached.

5. Four-particle correlated state in the core of an Abrikosov vortex

Analogous phenomena take place inside the core of a vortex in a conventional singlet type II superconductor, described by the complex scalar order-parameter field ψ, which in the vortex has the ordinary form (1.1). The same symmetry arguments may be applied, with the exception that now the generator $\hat{Q}$ of the continuous symmetry of the vortex is to be replaced by

$$\hat{Q} = \hat{L}_z - \tfrac{1}{2}\hat{I} \tag{5.1}$$

owing to the two-particle origin of the condensate in Fermi superfluids, i.e.

$$\hat{I}\psi = 2\psi \tag{5.2a}$$

$$\hat{U}(\Phi)\psi = \exp(2\mathrm{i}\Phi)\psi \tag{5.2b}$$

and the discrete combined symmetry $\hat{P}\hat{U}(\pi)$ in equation (2.10) should therefore be substituted by $\hat{P}\hat{U}(\pi/2)$.

This ordinary type of superconductivity is suppressed on the vortex axis owing to the phase winding; hence, the new type of superconductivity should dominate near the axis with the same symmetry. Salomaa and Volovik (1987) suggested that, if the spin–orbital coupling is essential in a superconductor, then a p-wave state similar to the A phase should arise on the vortex axis. In the strong-coupling limit, the generator $\hat{Q}$ of the continuous vortex symmetry becomes

$$\hat{Q}^{\text{spin-orbit}} = \hat{J}_z - \tfrac{1}{2}\hat{I} \tag{5.3}$$

where $\hat{J}_z$ is the total angular-momentum operator:

$$\hat{J}_z = \hat{S}_z + \hat{L}_z^{\text{int}} + \hat{L}_z^{\text{ext}}. \tag{5.4}$$

The core-matter state with $L_z^{\text{int}} = 1$, $S_z = 0$ and $L_z^{\text{ext}} = 0$ has the same $\hat{Q}$ symmetry as the vortex state in s-wave superconductors, with $L_z^{\text{int}} = 0$, $S_z = 0$, $L_z^{\text{ext}} = 1$. However, the discrete parity $\hat{P}\hat{U}(\pi/2)$ requires that L^{int} should be odd for this non-vanishing state, which means that the spin of the pairs of the core matter is to be in an eigenstate with $S = 1$, owing to the anti-symmetry of fermionic wavefunctions and, therefore, the spin–orbital interaction should be strong enough to produce a noticeable amplitude of the core spin-triplet state in a spin-singlet host superconductor.

If the spin–orbital interaction is small, then no pair-correlated state arises on the Abrikosov vortex axis. However, a different superconducting state should be formed in the vortex core in the manner discussed in §§ 2 and 3, but with the substitution $\hat{I} \to \hat{I}/2$. This means that the $I = 2$ state on the axis of the Bose-fluid vortex corresponds to the state with $I = 4$, with $\langle\hat{\psi}\hat{\psi}\hat{\psi}\hat{\psi}\rangle \neq 0$, on the axis of the singly quantised Abrikosov vortex, where the ordinary superconducting state is suppressed ($\langle\hat{\psi}\hat{\psi}\rangle = 0$). This is the spin-singlet ($S = 0$) four-particle-correlated ($I = 4$) superconducting state with $L_z^{\text{int}} = 2$ and $L_z^{\text{ext}} = 0$, which has $Q = 0$ according to equation (5.3), and the same discrete symmetries as the vortex in equation (1.1).

This state is also described by the traceless symmetric matrix B_{ij}, and all the Ginzburg–Landau analysis in § 3 is directly applicable also to the present situation. Note that, owing to the large core size of the Abrikosov vortex, the four-correlated superconductivity dominates on distances from the vortex axis which are still larger than the crystal lattice spacing, and therefore this core state should influence the observable change in the quasi-particle spectrum and in the density of states in the vortex-core matter.

6. Discussion

Many-body wavefunctions may be divided into classes distinguished by the manifolds of topologically charged zeros of the wavefunction; functions belonging into the same class may be continuously (adiabatically) transformed into each other via a smooth deformation of the manifold of zeros. The wavefunction for the vortex state of a conventional superfluid (or superconductor) is in the same class as the homogeneous superfluid state, but with an exotic d-wave *pair condensate* in the Bose liquid (or the homogeneous superconducting state with exotic d-wave *four-particle* condensate). This results in the existence of such novel states of exotic superfluidity (superconductivity) in the cores of the corresponding vortices.

The transformation inside the given class to some exotic many-particle-correlated states (which is adiabatic in nature, since the topological charge is not changed) may also occur under an external perturbation, including impurities. An example here is the formation of the isotropic four-particle-correlated superconducting state, with $\langle\hat{\psi}\hat{\psi}\hat{\psi}\hat{\psi}\rangle \neq 0$, in the anisotropic superconductor due to impurities (Volovik and Khmel'nitskiĭ 1984). Impurities suppress the anisotropic superconductivity; in the vicinity of the superconducting transition, the order parameter $\langle A\rangle \sim \langle\hat{\psi}\hat{\psi}\rangle = 0$, while $\langle AA\rangle \sim \langle\hat{\psi}\hat{\psi}\hat{\psi}\hat{\psi}\rangle$ remains non-zero owing to its isotropy. This mechanism was assumed to explain the observed (Smith *et al* 1984) splitting of the superconducting phase transition in the heavy-fermion compound UBe_{13} under doping by Th. It is not to be excluded that the observed splitting of the transition in the high-T_c superconductor Y–Ba–Cu–O (Indernees *et al* 1988, Ishikawa *et al* 1988a, b, Butera 1988) can also be related to the suppression of a highly anisotropic state by the presence of impurities in a narrow temperature interval in the vicinity of T_c, where only the more isotropic four-particle condensate survives.

It will be interesting to trace the general behaviour of a many-particle system when the topological charge of the zeros in the wavefunction increases and the system transforms stepwise from one class into another. Consequently, the state would evolve through classes of the intermediate types of superfluidity (with degrading off-diagonal long-range order) into classes of states which are normal but exhibit, for a given class, distinguishing properties, such as having, for example, a specific value for the Hall conductivity in the fractional QHE.

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